A Cosserat model of elastic solids reinforced by a family of curved and twisted fibers

David J. Steigmann and Milad Shirani

Department of Mechanical Engineering,
University of California, Berkeley,
CA. 94720, USA

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Elastic Energy and Kinematics

- Fibers are modelled as Kirchhoff Rods with referential directors $\{\boldsymbol{D}_1 = \boldsymbol{D}, \boldsymbol{D}_2, \boldsymbol{D}_3\}$ (shown in Figure 1), deformed directors $\{\boldsymbol{d}_1 = \boldsymbol{d}, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ with $\boldsymbol{D}$ and $\boldsymbol{d}$ being tangent to the fiber

$$\boldsymbol{D}_i \cdot \boldsymbol{D}_j = \delta_{ij} \qquad \text{and} \qquad \boldsymbol{d}_i \cdot \boldsymbol{d}_j = \delta_{ij} \tag{1}$$

- Rotation Tensor

$$\boldsymbol{R} = \boldsymbol{d} \otimes \boldsymbol{D} + \boldsymbol{d}_2 \otimes \boldsymbol{D}_2 + \boldsymbol{d}_3 \otimes \boldsymbol{D}_3 = \boldsymbol{d}_i \otimes \boldsymbol{D}_i \tag{2}$$

- Fibers are embedded in the body and they deform with the body. Therefore, they are material curves; so, $\boldsymbol{d}$ and $\boldsymbol{D}$ are material vectors and

$$|\,\boldsymbol{FD}\,|\,\boldsymbol{d} = \boldsymbol{FD} \tag{3}$$

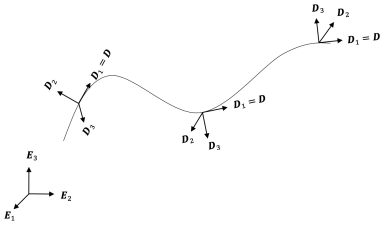

Figure 1: Schematic of a fiber in the reference configuration

Kinematical and constitutive variables in Cosserat elasticity

$$\boldsymbol{F}^{+}\boldsymbol{D} = \boldsymbol{F}^{-}\boldsymbol{D}, \quad \text{but} \quad \boldsymbol{F}^{+}\boldsymbol{D}_{\alpha} \neq \boldsymbol{F}^{-}\boldsymbol{D}_{\alpha}, \tag{4}$$

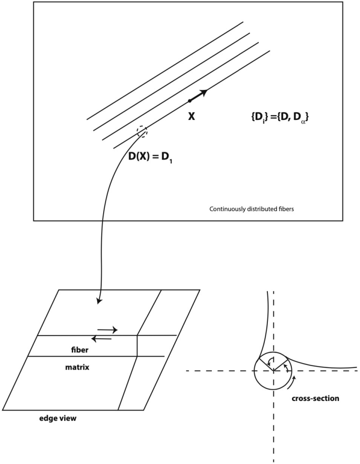

Figure 2: Fibers and matrix are kinematically independent; their interface convects as a material surface.

Elastic Energy and Kinematics (Rods)

$$\boldsymbol{r}'(s) = \lambda \boldsymbol{d}, \quad \text{and} \quad \lambda = \left|\boldsymbol{r}'(s)\right|, \tag{5}$$

$$\boldsymbol{R} = \boldsymbol{d}_i \otimes \boldsymbol{D}_i \qquad \text{with} \qquad \boldsymbol{D}_i(s) = \boldsymbol{A}(s)\boldsymbol{E}_i \tag{6}$$

$$\boldsymbol{d}_i' = \boldsymbol{R}'\boldsymbol{D}_i + \boldsymbol{R}\boldsymbol{D}_i' \tag{7}$$

$$\boldsymbol{d}_i' = \boldsymbol{W}\boldsymbol{d}_i = \boldsymbol{w} \times \boldsymbol{d}_i, \tag{8}$$

$$\boldsymbol{W} = \boldsymbol{R}'\boldsymbol{R}^t + \boldsymbol{R}\boldsymbol{A}'\boldsymbol{A}^t\boldsymbol{R}^t \tag{9}$$

$$\boldsymbol{w} = ax\boldsymbol{W} = \kappa_i \boldsymbol{d}_i, \tag{10}$$

$$\kappa_i = \frac{1}{2} e_{ijk} \boldsymbol{d}_k \cdot \boldsymbol{d}_j'. \tag{11}$$

Elastic Energy and Kinematics

$$S = \int_0^l U ds, \tag{12}$$

$$\{\boldsymbol{R}, \boldsymbol{R}', \boldsymbol{r}'\} \to \{\boldsymbol{QR}, \boldsymbol{QR}', \boldsymbol{Qr}'\} \qquad \text{select } \boldsymbol{Q} = \boldsymbol{R}^t_{|s} \tag{13}$$

$$\boldsymbol{R}^t \boldsymbol{W} \boldsymbol{R} - \boldsymbol{A}' \boldsymbol{A}^t = \boldsymbol{R}^t \boldsymbol{R}' = (\boldsymbol{R}\boldsymbol{D}_i \cdot \boldsymbol{R}' \boldsymbol{D}_j) \boldsymbol{D}_i \otimes \boldsymbol{D}_j, \tag{14}$$

$$\boldsymbol{\gamma} = \gamma_i \boldsymbol{D}_i = ax(\boldsymbol{R}^t \boldsymbol{R}') \qquad \text{with} \qquad \gamma_i = \frac{1}{2} e_{ijk} \boldsymbol{R}\boldsymbol{D}_k \cdot \boldsymbol{R}' \boldsymbol{D}_j. \tag{15}$$

$$U = w(\lambda, \boldsymbol{\gamma}; s). \tag{16}$$

$$\gamma_i = \kappa_i - \kappa_i^0, \qquad \text{with} \qquad \kappa_i^0 = \frac{1}{2} e_{ijk} \boldsymbol{D}_k \cdot \boldsymbol{D}_j' \tag{17}$$

$$\boldsymbol{\gamma} = \boldsymbol{\kappa} - \boldsymbol{\kappa}^0, \tag{18}$$

$$\boldsymbol{\kappa} = \boldsymbol{R}^t \boldsymbol{w} = \kappa_i \boldsymbol{D}_i = ax(\boldsymbol{R}^t \boldsymbol{W} \boldsymbol{R}). \tag{19}$$

$$w(\lambda, \boldsymbol{\kappa}; s) = \frac{1}{2} A(s) \varepsilon^2 + \frac{1}{2} T(s) \tau^2 + \frac{1}{2} F(s) \kappa_\alpha \kappa_\alpha, \tag{20}$$

Equilibrium

$$\dot{S} = P \qquad \text{where} \qquad S = \int_0^l U ds, \tag{21}$$

$$\dot{U} = \dot{w} = w_\lambda \dot{\lambda} + m_i \dot{\gamma}_i, \qquad \text{with} \qquad w_\lambda = \partial w / \partial \lambda \quad \text{and} \quad m_i = \partial w / \partial \gamma_i \tag{22}$$

$$\lambda \boldsymbol{d} = \boldsymbol{F}\boldsymbol{D} \Longrightarrow \dot{\lambda} \boldsymbol{d} + \boldsymbol{\omega} \times \boldsymbol{r}' = \boldsymbol{u}' \qquad \text{and} \qquad \dot{\boldsymbol{d}}_i = \boldsymbol{\omega} \times \boldsymbol{d}_i. \tag{23}$$

$$\dot{\kappa}_i = \frac{1}{2} e_{ijk} (\dot{\boldsymbol{d}}_k \cdot \boldsymbol{d}_j' + \boldsymbol{d}_k \cdot \dot{\boldsymbol{d}}_j') = \frac{1}{2} e_{ijk} [\boldsymbol{\omega} \times \boldsymbol{d}_k \cdot \boldsymbol{d}_j' + \boldsymbol{d}_k \cdot (\boldsymbol{\omega}' \times \boldsymbol{d}_j + \boldsymbol{\omega} \times \boldsymbol{d}_j')] \tag{24}$$

$$\dot{\gamma}_i = \dot{\kappa}_i = \boldsymbol{d}_i \cdot \boldsymbol{\omega}'. \tag{25}$$

$$\dot{S} = \int_0^l (w_\lambda \boldsymbol{d} \cdot \boldsymbol{u}' + \boldsymbol{m} \cdot \boldsymbol{\omega}') ds, \tag{26}$$

$$\boldsymbol{m} = m_i \boldsymbol{d}_i. \tag{27}$$

$$\boldsymbol{r}' \cdot \boldsymbol{d}_\alpha = 0; \quad \alpha = 2, 3. \qquad \text{constraints} \tag{28}$$

$$E = S + \int_0^l f_\alpha \boldsymbol{r}' \cdot \boldsymbol{d}_\alpha ds, \qquad f_\alpha(s) \text{ are Lagrange multipliers} \tag{29}$$

$$\dot{E} = P, \tag{30}$$

where

$$\dot{E} = \int_0^l [(w_\lambda \boldsymbol{d} + f_\alpha \boldsymbol{d}_\alpha) \cdot \boldsymbol{u}' + \boldsymbol{m} \cdot \boldsymbol{\omega}' + f_\alpha \boldsymbol{d}_\alpha \times \boldsymbol{r}' \cdot \boldsymbol{\omega} + \dot{f}_\alpha \boldsymbol{r}' \cdot \boldsymbol{d}_\alpha] ds \tag{31}$$

$$\dot{E} = (\boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{m} \cdot \boldsymbol{\omega}) \,|_0^l - \int_0^l [\boldsymbol{u} \cdot \boldsymbol{f}' + \boldsymbol{\omega} \cdot (\boldsymbol{m}' - \boldsymbol{f} \times \boldsymbol{r}')] ds, \tag{32}$$

$$\boldsymbol{f} = w_\lambda \boldsymbol{d} + f_\alpha \boldsymbol{d}_\alpha. \tag{33}$$

$$P = (\boldsymbol{t} \cdot \boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega}) \,|_0^l + \int_0^l (\boldsymbol{u} \cdot \boldsymbol{g} + \boldsymbol{\omega} \cdot \boldsymbol{\pi}) ds, \tag{34}$$

$$\boldsymbol{f}' + \boldsymbol{g} = 0 \quad \text{and} \quad \boldsymbol{m}' + \boldsymbol{\pi} = \boldsymbol{f} \times \boldsymbol{r}', \quad \text{endpoint conditions are} \quad \boldsymbol{f} = \boldsymbol{t} \quad \text{and} \quad \boldsymbol{m} = \boldsymbol{c} \tag{35}$$

Equilibrium

From the strain-energy function

$$w(\lambda, \boldsymbol{\kappa}; s) = \frac{1}{2}A(s)\varepsilon^2 + \frac{1}{2}T(s)\tau^2 + \frac{1}{2}F(s)\kappa_\alpha\kappa_\alpha, \tag{36}$$

we find

$$m_1\boldsymbol{d}_1 = T\tau\boldsymbol{d} \qquad \text{and} \qquad m_\alpha\boldsymbol{d}_\alpha = F\kappa_\alpha\boldsymbol{d}_\alpha \tag{37}$$

Moreover,

$$\kappa_i = \frac{1}{2}e_{ijk}\boldsymbol{d}_k \cdot \boldsymbol{d}_j' \qquad \text{with} \qquad \boldsymbol{d} \cdot \boldsymbol{d}_\mu' = -\boldsymbol{d}_\mu \cdot \boldsymbol{d} \Longrightarrow \kappa_\alpha = e_{\alpha 1\mu}\boldsymbol{d}_\mu \cdot \boldsymbol{d}'. \tag{38}$$

Also $\boldsymbol{d} \cdot \boldsymbol{d}' = 0$; therefore

$$\boldsymbol{d}' = (\boldsymbol{d}_\alpha \cdot \boldsymbol{d}')\boldsymbol{d}_\alpha \quad \text{and} \qquad \boldsymbol{d} \times \boldsymbol{d}' = (\boldsymbol{d}_\alpha \cdot \boldsymbol{d}')\boldsymbol{d} \times \boldsymbol{d}_\alpha = (e_{\beta 1\alpha}\boldsymbol{d}_\alpha \cdot \boldsymbol{d}')\boldsymbol{d}_\beta \tag{39}$$

thus

$$\kappa_\beta\boldsymbol{d}_\beta = \boldsymbol{d} \times \boldsymbol{d}' \longrightarrow m_\alpha\boldsymbol{d}_\alpha = F\kappa_\alpha\boldsymbol{d}_\alpha = F\boldsymbol{d} \times \boldsymbol{d}' \tag{40}$$

so

$$\boldsymbol{m} = T\tau\boldsymbol{d} + F\boldsymbol{d} \times \boldsymbol{d}'. \tag{41}$$

Kinematical and constitutive variables in Cosserat elasticity

$$\boldsymbol{FD} = \lambda \boldsymbol{d}, \quad \text{where} \quad \boldsymbol{d} = \boldsymbol{RD} \quad \text{and} \quad \lambda = |\boldsymbol{FD}|\,, \tag{42}$$

$$\boldsymbol{D}_\alpha \cdot \boldsymbol{R}^t \boldsymbol{FD} = 0; \quad \alpha = 2,3, \qquad \text{constraints} \tag{43}$$

$$U(\boldsymbol{F}, \boldsymbol{R}, \nabla \boldsymbol{R}; \boldsymbol{X}), \tag{44}$$

$$\boldsymbol{F} = F_{iA} \boldsymbol{e}_i \otimes \boldsymbol{E}_A, \quad \boldsymbol{R} = R_{iA} \boldsymbol{e}_i \otimes \boldsymbol{E}_A \quad \text{and} \quad \nabla \boldsymbol{R} = R_{iA,B} \boldsymbol{e}_i \otimes \boldsymbol{E}_A \otimes \boldsymbol{E}_B \quad \text{with} \quad F_{iA} = \chi_{i,A}, \tag{45}$$

$$U(\boldsymbol{F}, \boldsymbol{R}, \nabla \boldsymbol{R}; \boldsymbol{X}) = U(\boldsymbol{QF}, \boldsymbol{QR}, \boldsymbol{Q}\nabla \boldsymbol{R}; \boldsymbol{X}) = U(\boldsymbol{R}^T \boldsymbol{F}, \boldsymbol{R}^T \nabla \boldsymbol{R}; \boldsymbol{X}) = W(\boldsymbol{E}, \boldsymbol{\Gamma}; \boldsymbol{X}), \tag{46}$$

$$\boldsymbol{E} = \boldsymbol{R}^t \boldsymbol{F} = E_{AB} \boldsymbol{E}_A \otimes \boldsymbol{E}_B; \qquad E_{AB} = R_{iA} F_{iB}, \tag{47}$$

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_C \otimes \boldsymbol{E}_C = \Gamma_{DC} \boldsymbol{E}_D \otimes \boldsymbol{E}_C; \qquad \Gamma_{DC} = \frac{1}{2} e_{BAD} R_{iA} R_{iB,C}, \tag{48}$$

Equilibrium

$$\dot{E} = P, \tag{49}$$

$$E = \int_\kappa U dv + \int_\kappa \Lambda_\alpha \boldsymbol{D}_\alpha \cdot \boldsymbol{E}\boldsymbol{D} dv, \quad (\Lambda_\alpha \text{ are Lagrange multipliers}) \tag{50}$$

$$\dot{U} = \dot{W} = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{E}} + \boldsymbol{\mu} \cdot \dot{\boldsymbol{\Gamma}}, \qquad \text{with} \qquad \boldsymbol{\sigma} = W_{\boldsymbol{E}} \quad \text{and} \quad \boldsymbol{\mu} = W_{\boldsymbol{\Gamma}} \tag{51}$$

$$\dot{E} = \int_\kappa [(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \cdot \dot{\boldsymbol{E}} + \boldsymbol{\mu} \cdot \dot{\boldsymbol{\Gamma}} + \dot{\Lambda}_\alpha \boldsymbol{D}_\alpha \cdot \boldsymbol{E}\boldsymbol{D}] dv \quad \text{where} \quad \boldsymbol{\Lambda} = \Lambda_\alpha \boldsymbol{D}_\alpha \tag{52}$$

$$\dot{\boldsymbol{E}} = \boldsymbol{R}^t (\nabla \boldsymbol{u} - \boldsymbol{\Omega} \boldsymbol{F}), \quad \text{where} \quad \boldsymbol{u} = \dot{\boldsymbol{\chi}} \quad \text{and} \quad \boldsymbol{\Omega} = \dot{\boldsymbol{R}} \boldsymbol{R}^t. \tag{53}$$

$$(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \cdot \dot{\boldsymbol{E}} = \boldsymbol{R}(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\Omega} \cdot Skw[\boldsymbol{R}(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \boldsymbol{F}^t] \tag{54}$$

$$(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \cdot \dot{\boldsymbol{E}} = \boldsymbol{R}(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \cdot \nabla \boldsymbol{u} - 2ax\{\boldsymbol{R} Skw[(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D}) \boldsymbol{E}^t] \boldsymbol{R}^t\} \cdot \boldsymbol{\omega} \tag{55}$$

$$\dot{\boldsymbol{\Gamma}} = \boldsymbol{R}^t \nabla \boldsymbol{\omega} \Longrightarrow \boldsymbol{\mu} \cdot \dot{\boldsymbol{\Gamma}} = \boldsymbol{R}\boldsymbol{\mu} \cdot \nabla \boldsymbol{\omega}, \tag{56}$$

In detail

We have $\boldsymbol{E} = \boldsymbol{R}^T\boldsymbol{F}$ and $\Gamma_{DC} = \frac{1}{2}e_{BAD}R_{iA}R_{iB,C}$, thus

$$\begin{aligned}\boldsymbol{E} = \boldsymbol{R}^T\boldsymbol{F} \Longrightarrow \dot{\boldsymbol{E}} &= \dot{\boldsymbol{R}}^T\boldsymbol{F} + \boldsymbol{R}^T\dot{\boldsymbol{F}} \\ &= (\boldsymbol{\Omega}\boldsymbol{R})^T\boldsymbol{F} + \boldsymbol{R}^T\nabla\boldsymbol{u} \\ &= \boldsymbol{R}^T(\nabla\boldsymbol{u} - \boldsymbol{\Omega}\boldsymbol{F}) \end{aligned} \tag{57}$$

similarly

$$\begin{aligned}\dot{\Gamma}_{DC} &= \frac{1}{2}e_{BAD}(\dot{R}_{iA}R_{iB,C} + R_{iA}\dot{R}_{iB,C}) \\ &= \frac{1}{2}e_{BAD}(\Omega_{im}R_{mA}R_{iB,C} + R_{iA}\Omega_{im,C}R_{mB} + R_{iA}\Omega_{im}R_{mB,C}) \\ &= \frac{1}{2}e_{BAD}\left\{\Omega_{im}(R_{mA}R_{iB,C} + R_{iA}R_{mB,C}) + R_{iA}\Omega_{im,C}R_{mB}\right\} \\ &= \frac{1}{2}(e_{BAD}R_{iA}R_{mB})\Omega_{im,C} \\ &= \frac{1}{2}(e_{mij}\Omega_{im,C})R_{jD} \qquad \text{(because } e_{mij}R_{jD} = e_{BAD}R_{iA}R_{mB}) \\ &= (\frac{1}{2}e_{mij}\Omega_{im})_{,C}R_{jD} \qquad \text{(because } \omega_j = \frac{1}{2}e_{mij}\Omega_{im}) \\ &= R_{jD}\omega_{j,C} \end{aligned} \tag{58}$$

$$\Longrightarrow \dot{\boldsymbol{\Gamma}} = \boldsymbol{R}^T\nabla\boldsymbol{\omega} \tag{59}$$

Equilibrium

$$
\begin{aligned}
\dot{E} \quad = \quad & \int_{\partial\kappa} [(\boldsymbol{R}\boldsymbol{\sigma} + \boldsymbol{\lambda} \otimes \boldsymbol{D})\boldsymbol{\nu} \cdot \boldsymbol{u} + (\boldsymbol{R}\boldsymbol{\mu})\boldsymbol{\nu} \cdot \boldsymbol{\omega}] da + \int_{\kappa} \dot{\Lambda}_\alpha \boldsymbol{D}_\alpha \cdot \boldsymbol{E}\boldsymbol{D} dv \\
& - \int_{\kappa} \{\boldsymbol{u} \cdot Div(\boldsymbol{R}\boldsymbol{\sigma} + \boldsymbol{\lambda} \otimes \boldsymbol{D})\} dv \qquad\qquad (\boldsymbol{\lambda} = \boldsymbol{R}\boldsymbol{\Lambda} = \Lambda_\alpha \boldsymbol{d}_\alpha) \\
& - \int_{\kappa} \{\boldsymbol{\omega} \cdot [Div(\boldsymbol{R}\boldsymbol{\mu}) + 2ax(\boldsymbol{R} Skw[(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D})\boldsymbol{E}^t]\boldsymbol{R}^t)]\} dv,
\end{aligned} \tag{60}
$$

$$
P = \int_{\partial\kappa} (\boldsymbol{t} \cdot \boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega}) da + \int_{\kappa} (\boldsymbol{g} \cdot \boldsymbol{u} + \boldsymbol{\pi} \cdot \boldsymbol{\omega}) dv, \tag{61}
$$

where $\boldsymbol{t}$ and $\boldsymbol{c}$ are densities of force and couple acting on $\partial\kappa$, and $\boldsymbol{g}$ and $\boldsymbol{\pi}$ are densities of force and couple acting in κ.

$$
\boldsymbol{g} = -Div(\boldsymbol{R}\boldsymbol{\sigma} + \boldsymbol{\lambda} \otimes \boldsymbol{D}) \quad \text{and} \quad \boldsymbol{\pi} = -Div(\boldsymbol{R}\boldsymbol{\mu}) - 2ax\{\boldsymbol{R} Skw[(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \boldsymbol{D})\boldsymbol{E}^t]\boldsymbol{R}^t\} \quad \text{in} \quad \kappa, \tag{62}
$$

and the natural boundary conditions

$$
\boldsymbol{t} = (\boldsymbol{R}\boldsymbol{\sigma} + \boldsymbol{\lambda} \otimes \boldsymbol{D})\boldsymbol{\nu} \quad \text{on} \quad \partial\kappa_t \quad \text{and} \quad \boldsymbol{c} = (\boldsymbol{R}\boldsymbol{\mu})\boldsymbol{\nu} \quad \text{on} \quad \partial\kappa_c, \tag{63}
$$

where $\partial\kappa_t$ is a part of $\partial\kappa$ where position is not assigned and $\partial\kappa_c$ is a part where rotation is not assigned. We assume position to be assigned on $\partial\kappa \setminus \partial\kappa_t$ ($\boldsymbol{u} = 0$), and rotation to be assigned on $\partial\kappa \setminus \partial\kappa_c$ ($\boldsymbol{\omega} = 0$).

Fiber-matrix interaction

$$\boldsymbol{\gamma} = ax(\boldsymbol{R}^t\boldsymbol{R}') = \gamma_i \boldsymbol{D}_i \quad \text{with} \quad \gamma_i = \frac{1}{2} e_{ijk} \boldsymbol{D}_k \cdot \boldsymbol{R}^t \boldsymbol{R}' \boldsymbol{D}_j \qquad \text{and} \qquad \boldsymbol{R} = \boldsymbol{d}_i \otimes \boldsymbol{D}_i \quad (64)$$

$$\boldsymbol{R}^t\boldsymbol{R}' = R_{iC} R_{iA,B} D_B \boldsymbol{E}_C \otimes \boldsymbol{E}_A = e_{ACD} \Gamma_{DB} D_B \boldsymbol{E}_C \otimes \boldsymbol{E}_A \quad (65)$$

$$\Longrightarrow \gamma = \Gamma \boldsymbol{D} \quad (66)$$

$$W(\boldsymbol{E}, \boldsymbol{\Gamma}; \boldsymbol{X}) = w(\boldsymbol{E}, \gamma; \boldsymbol{X}), \quad (67)$$

$$\boldsymbol{\sigma} = w_{\boldsymbol{E}}. \quad (68)$$

$$\dot{\gamma}_i = \boldsymbol{R}\boldsymbol{D}_i \cdot (\nabla \boldsymbol{\omega}) \boldsymbol{D} = \boldsymbol{D}_i \cdot (\boldsymbol{R}^t \nabla \boldsymbol{\omega}) \boldsymbol{D} = \boldsymbol{D}_i \otimes \boldsymbol{D} \cdot \dot{\boldsymbol{\Gamma}} \quad (69)$$

$$\boldsymbol{\mu} \cdot \dot{\boldsymbol{\Gamma}} = \dot{W} = \dot{w} = w_{\gamma} \cdot \dot{\gamma} = \boldsymbol{M} \otimes \boldsymbol{D} \cdot \dot{\boldsymbol{\Gamma}}, \quad (70)$$

$$\boldsymbol{M} = w_{\gamma} = m_i \boldsymbol{D}_i \quad \text{with} \quad m_i = \partial w / \partial \gamma_i, \quad (71)$$

$$\boldsymbol{\mu} = \boldsymbol{M} \otimes \boldsymbol{D}. \quad (72)$$

Fiber-matrix interaction

$$Div(\boldsymbol{\lambda}\otimes\boldsymbol{D}) = \boldsymbol{\lambda}' + (Div\boldsymbol{D})\boldsymbol{\lambda}, \quad \text{where} \quad \boldsymbol{\lambda}' = (\nabla\boldsymbol{\lambda})\boldsymbol{D} \tag{73}$$

$$\begin{aligned} 2ax\{\boldsymbol{R}Skw[(\boldsymbol{\Lambda}\otimes\boldsymbol{D})\boldsymbol{E}^t]\boldsymbol{R}^t\} &= 2ax\{Skw[\boldsymbol{R}(\boldsymbol{\Lambda}\otimes\boldsymbol{D})\boldsymbol{E}^t\boldsymbol{R}^t]\} \\ &= 2ax[Skw(\boldsymbol{\lambda}\otimes\boldsymbol{F}\boldsymbol{D})] \\ &= ax(\boldsymbol{\lambda}\otimes\boldsymbol{\chi}' - \boldsymbol{\chi}'\otimes\boldsymbol{\lambda}) \\ &= \boldsymbol{\chi}'\times\boldsymbol{\lambda}, \qquad \text{where} \qquad \boldsymbol{\chi}' = (\nabla\boldsymbol{\chi})\boldsymbol{D} \end{aligned} \tag{74}$$

with

$$Div(\boldsymbol{R}\boldsymbol{\mu}) = \boldsymbol{m}' + (Div\boldsymbol{D})\boldsymbol{m}, \quad \text{where} \quad \boldsymbol{m} = \boldsymbol{R}\boldsymbol{M} = m_i\boldsymbol{d}_i, \tag{75}$$

we find

$$\boldsymbol{\lambda}' + (Div\boldsymbol{D})\boldsymbol{\lambda} + Div(\boldsymbol{R}\boldsymbol{\sigma}) + \boldsymbol{g} = 0, \quad \boldsymbol{t} = (\boldsymbol{R}\boldsymbol{\sigma})\boldsymbol{\nu} + (\boldsymbol{D}\cdot\boldsymbol{\nu})\boldsymbol{\lambda} \tag{76}$$

and

$$\boldsymbol{m}' + \boldsymbol{\chi}'\times\boldsymbol{\lambda} + (Div\boldsymbol{D})\boldsymbol{m} + 2ax[\boldsymbol{R}Skw(\boldsymbol{\sigma}\boldsymbol{E}^t)\boldsymbol{R}^t] + \boldsymbol{\pi} = 0, \qquad \boldsymbol{c} = (\boldsymbol{D}\cdot\boldsymbol{\nu})\boldsymbol{m} \tag{77}$$

Some Remarks

- Fiber inextensibility is accommodated by appending the constraint $\boldsymbol{D} \cdot \boldsymbol{E}\boldsymbol{D} = 1$;
- In this case $\boldsymbol{\Lambda}$ and $\boldsymbol{\lambda}$ are now 3-vectors given respectively by $\Lambda_i \boldsymbol{D}_i$ and $\Lambda_i \boldsymbol{d}_i$ in which Λ_1 is a constitutively undetermined density of axial force exerted on the fibers.
- Incompressibility entails the constraint $\det \boldsymbol{F} (= \det \boldsymbol{E}) = 1$, which may be accommodated by using

$$\bar{W} = W + \Lambda_\alpha \boldsymbol{D}_\alpha \cdot \boldsymbol{E}\boldsymbol{D} - p(\det \boldsymbol{E} - 1) \qquad p \text{ is Lagrange multiplier} \tag{78}$$

and we find

$$Div(\boldsymbol{R}\boldsymbol{\sigma} - p\boldsymbol{F}^* + \boldsymbol{\lambda} \otimes \boldsymbol{D}) = 0 \quad \text{and} \quad \boldsymbol{t} = (\boldsymbol{R}\boldsymbol{\sigma} - p\boldsymbol{F}^* + \boldsymbol{\lambda} \otimes \boldsymbol{D})\boldsymbol{n}, \tag{79}$$

augmented by the Piola identity $Div \boldsymbol{F}^* = 0$.

The dependence of the strain-energy function on γ (or Γ) introduces a natural length scale, L say, into the constitutive theory which is on the order of that of the microstructure and hence of the diameter of a fiber cross section or the spacing between adjacent fibers. Using the larger of these to define the dimensionless curvature-twist vector $L\boldsymbol{\gamma}$, supposing that $|L\boldsymbol{\gamma}| \ll 1$ in typical applications and assuming that the fibers transmit no moments when γ vanishes, we find that w is given to leading order by

$$w(\boldsymbol{E}, \gamma; \boldsymbol{X}) = \varpi(\boldsymbol{E}; \boldsymbol{X}) + \frac{1}{2}\gamma \cdot \boldsymbol{K}(\boldsymbol{E}; \boldsymbol{X})\gamma, \tag{80}$$

where

$$\varpi(\boldsymbol{E}; \boldsymbol{X}) = w(\boldsymbol{E}, 0; \boldsymbol{X}) \qquad \text{and} \qquad \boldsymbol{K}(\boldsymbol{E}; \boldsymbol{X}) = w_{\gamma\gamma|\gamma=0} \tag{81}$$

For $\boldsymbol{E}$ close to $\boldsymbol{I}$ we have $\boldsymbol{K}(\boldsymbol{E}; \boldsymbol{X}) = \boldsymbol{K}(\boldsymbol{I}; \boldsymbol{X}) + O(|\boldsymbol{E} - \boldsymbol{I}|)$, provided that $\boldsymbol{K}(\cdot; \boldsymbol{X})$ is differentiable. Then the energy is approximated, as in

$$w(\lambda, \gamma; s) = \frac{1}{2}A(s)\varepsilon^2 + \frac{1}{2}T(s)\tau^2 + \frac{1}{2}F(s)\gamma_\alpha\kappa_\alpha \qquad \text{with} \qquad \varepsilon = \lambda - 1 \tag{82}$$

by the decoupled energy

$$w(\boldsymbol{E}, \gamma; \boldsymbol{X}) = \varpi(\boldsymbol{E}; \boldsymbol{X}) + \varphi(\gamma; \boldsymbol{X}), \tag{83}$$

for some homogeneous quadratic function $\varphi(\cdot; \boldsymbol{X})$.

An Example: Torsion of a cylinder

The reference placement κ of the body in the cylindrical polar coordinate system (r, θ, z) is the region defined by

$$0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq L \tag{84}$$

Position of a material point in the reference configuration and deformed configuration is

$$\boldsymbol{X} = r\boldsymbol{e}_r(\theta) + z\boldsymbol{k}, \qquad \chi(\boldsymbol{X}) = r\boldsymbol{e}_r(\phi) + z\boldsymbol{k} \qquad \text{where} \qquad \phi = \theta + \tau z \tag{85}$$

The fiber are assumed to be everywhere aligned with the axis of the cylinder in the reference placement; thus, $\boldsymbol{D} = \boldsymbol{k}$, the fiber derivative is $(\cdot)' = \partial(\cdot)/\partial z$

Consider the elastic energy of the form

$$w(\boldsymbol{E}, \boldsymbol{\gamma}) = w_1(\boldsymbol{E}) + w_2(\boldsymbol{E})(\boldsymbol{\gamma} \cdot \boldsymbol{D})^2 + w_3(\boldsymbol{E}) \mid 1\boldsymbol{\gamma} \mid^2 \tag{86}$$

and

$$w_1(\boldsymbol{E}) = \frac{1}{2}\mu(I_1 - 3), \quad w_2(\boldsymbol{E}) = \frac{1}{2}T \quad w_3(\boldsymbol{E}) = \frac{1}{2}F \qquad \text{and} \quad 1 = \boldsymbol{I} - \boldsymbol{D} \otimes \boldsymbol{D} \tag{87}$$

where $I_1 = tr(\boldsymbol{E}^T\boldsymbol{E})$, μ, T and F are positive constants. Thus,

$$\boldsymbol{\sigma} = \mu\boldsymbol{E} \quad \text{and} \quad \boldsymbol{m} = T\gamma\boldsymbol{d} + F\boldsymbol{d} \times \boldsymbol{d}' \quad \text{where} \quad \gamma = \boldsymbol{\gamma} \cdot \boldsymbol{D} \tag{88}$$

An Example: Torsion of a cylinder

By substituting the response functions in the balance laws, we find

$$\mu \mathrm{div} \boldsymbol{B} + \boldsymbol{\lambda}' = \mathrm{grad} p \qquad \text{and} \qquad \boldsymbol{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} = 0 \tag{89}$$

Consider deformations that satisfy

$$\mu \mathrm{div} \boldsymbol{B} = \mathrm{grad} p \Longrightarrow \boldsymbol{\lambda}' = 0 \tag{90}$$

Deformation gradient

$$d\boldsymbol{\chi} = \boldsymbol{F} d\boldsymbol{X} \Longrightarrow \boldsymbol{F} = \boldsymbol{Q}[\boldsymbol{I} + r\tau \boldsymbol{e}_\theta \otimes \boldsymbol{k}] \tag{91}$$

with

$$\boldsymbol{Q} = \boldsymbol{e}_r(\phi) \otimes \boldsymbol{e}_r(\theta) + \boldsymbol{e}_\theta(\phi) \otimes \boldsymbol{e}_\theta(\theta) + \boldsymbol{k} \otimes \boldsymbol{k} \in \mathrm{Orth}^+ \tag{92}$$

This deformation is isochoric and hence kinematically admissible in an incompressible material.

Also

$$\lambda \boldsymbol{d} = \boldsymbol{F}\boldsymbol{D} = \boldsymbol{k} + r\tau \boldsymbol{e}_\theta(\phi) \qquad \text{and} \qquad \lambda = \sqrt{1 + r^2\tau^2} \tag{93}$$

We have

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{I} + r\tau[\boldsymbol{e}_\theta(\phi) \otimes \boldsymbol{k} + \boldsymbol{k} \otimes \boldsymbol{e}_\theta(\phi)] + r^2\tau^2 \boldsymbol{e}_\theta(\phi) \otimes \boldsymbol{e}_\theta(\phi) \tag{94}$$

and

$$\mathrm{div} \boldsymbol{B} = -r\tau^2 \boldsymbol{e}_r(\phi) \longrightarrow \mathrm{grad} p = -\frac{r\tau^2}{\mu} \boldsymbol{e}_r(\phi) \longrightarrow p(r) = p_0 - \frac{1}{2}\mu\tau^2 r^2 \tag{95}$$

where p_0 is a constant.

An Example: Torsion of a cylinder

We have

$$\boldsymbol{m} = T\gamma \boldsymbol{d} + F\boldsymbol{d} \times \boldsymbol{d}' \longrightarrow \boldsymbol{m}' = T\gamma' \boldsymbol{d} + T\gamma \boldsymbol{d}' + F\boldsymbol{d} \times \boldsymbol{d}'' \tag{96}$$

and, because

$$\boldsymbol{\lambda} = \boldsymbol{R}\boldsymbol{\Lambda} = \Lambda_\alpha \boldsymbol{d}_\alpha \longrightarrow \boldsymbol{\lambda} \cdot \boldsymbol{d} = 0 \tag{97}$$

we find

$$\boldsymbol{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} = 0 \xrightarrow{\cdot \boldsymbol{d}} \boldsymbol{m}' \cdot \boldsymbol{d} = 0 \Longrightarrow \gamma' = (\boldsymbol{\gamma} \cdot \boldsymbol{D})' = 0 \tag{98}$$

Also

$$\lambda \boldsymbol{d} = \boldsymbol{F}\boldsymbol{D} = \boldsymbol{k} + r\tau \boldsymbol{e}_\theta(\phi) \longrightarrow \lambda \boldsymbol{d}' = -r\tau^2 \boldsymbol{e}_r(\phi) \quad \text{and} \quad \boldsymbol{d} \times \boldsymbol{d}'' = \lambda^{-2} r\tau^3 \boldsymbol{e}_r \tag{99}$$

so

$$\boldsymbol{m}' = \frac{r\tau^2}{\lambda}\left(\frac{F\tau}{\lambda} - T\gamma\right)\boldsymbol{e}_r(\phi) \tag{100}$$

Moreover,

$$\boldsymbol{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} = 0 \xrightarrow{\boldsymbol{d}\times} \boldsymbol{\lambda} = \frac{1}{\lambda}\boldsymbol{d} \times \boldsymbol{m}' = -\frac{r\tau^2}{\lambda^3}\left(\frac{F\tau}{\lambda} - T\gamma\right)[r\tau \boldsymbol{k} - \boldsymbol{e}_\theta(\phi)] \tag{101}$$

and

$$\boldsymbol{\lambda}' = 0, \quad \text{and} \quad \boldsymbol{e}_\theta' = -\tau \boldsymbol{e}_r(\phi) \longrightarrow \gamma = \frac{F\tau}{\lambda T} \tag{102}$$

and as a result

$$\boldsymbol{m} = F\tau \boldsymbol{k} \tag{103}$$

implying that every fiber transmits the same moment. This result is interesting in light of the fact that the individual terms in $\boldsymbol{m} = T\gamma \boldsymbol{d} + F\boldsymbol{d} \times \boldsymbol{d}'$ associated with fiber twisting and bending are non-uniform.

An Example: Torsion of a cylinder

To complete the solution we impose the traction condition
($\boldsymbol{t} = (\boldsymbol{R\sigma} - p\boldsymbol{F}^* + \boldsymbol{\lambda} \otimes \boldsymbol{D})\boldsymbol{n}$, with $\boldsymbol{D} = \boldsymbol{k}$)

$$(\boldsymbol{R\sigma} - p\boldsymbol{F}^*)\boldsymbol{e}_r(\theta) = 0 \quad \text{at} \quad r = a. \tag{104}$$

This is equivalent to $(\boldsymbol{R\sigma})\boldsymbol{F}^t\boldsymbol{e}_r(\phi) = p\boldsymbol{e}_r(\phi)$ and thus, in the present circumstances, to

$$\mu\boldsymbol{B}\boldsymbol{e}_r(\phi) = p\boldsymbol{e}_r(\phi) \quad \text{at} \quad r = a, \tag{105}$$

furnishing $p(a) = \mu$ and hence $p(r) = \frac{1}{2}\mu\tau^2(a^2 - r^2) + \mu$, finally yielding

$$(\boldsymbol{R\sigma})\boldsymbol{F}^t - p\boldsymbol{I} = \mu[\frac{1}{2}\tau^2(r^2 - a^2) - 1]\boldsymbol{I} + \mu\boldsymbol{B}. \tag{106}$$

The overall response of the cylinder may be determined by computing the net force on a cross section and the net torque required to effect the torsion. These in turn require the traction

$$\boldsymbol{t} = [(\boldsymbol{R\sigma})\boldsymbol{F}^t - p\boldsymbol{I}]\boldsymbol{k} = \frac{1}{2}\mu\tau^2(r^2 - a^2)\boldsymbol{k} + \mu r\tau\boldsymbol{e}_\theta(\phi) \tag{107}$$

acting on a cross section. This is the same as the traction appearing in

$$\boldsymbol{t} = (\boldsymbol{R\sigma} + \boldsymbol{\lambda} \otimes \boldsymbol{D})\boldsymbol{n} \tag{108}$$

because there is no change in cross-sectional area in the course of the deformation.

The resultant force is

$$\boldsymbol{f} = \int_0^{2\pi} \int_0^a \boldsymbol{t} r dr d\phi = f(\tau)\boldsymbol{k}, \tag{109}$$

where

$$f(\tau) = -\frac{1}{4}\pi a^4 \mu \tau^2, \tag{110}$$

and is a manifestation of the well known normal-stress effect in nonlinear elasticity.

Finally, the torque is

$$\boldsymbol{\rho} = \int_0^{2\pi} \int_0^a (\boldsymbol{\chi} \times \boldsymbol{t} + \boldsymbol{m}) r dr d\phi = \rho(\tau)\boldsymbol{k}, \tag{111}$$

where

$$\rho(\tau) = \pi a^2 \tau (F + \frac{1}{2}\mu a^2). \tag{112}$$

An Example: Flexure

To describe flexure of a rectangular block we use Cartesian coordinates in the reference placement and polars in the current placement. Specifically,

$$\boldsymbol{X} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \quad \text{and} \quad \chi(\boldsymbol{X}) = r\boldsymbol{e}_r(\theta) + z\boldsymbol{k}, \quad \text{where} \quad r = f(x) \quad \text{and} \quad \theta = g(y), \tag{113}$$

for some functions f and g, to be determined. The deformation gradient is

$$\boldsymbol{F} = f'\boldsymbol{e}_r \otimes \boldsymbol{i} + fg'\boldsymbol{e}_\theta \otimes \boldsymbol{j} + \boldsymbol{k} \otimes \boldsymbol{k}, \tag{114}$$

yielding

$$1 = J = f(x)f'(x)g'(y) \tag{115}$$

in the case of incompressibility. This implies that $g' = C$, a constant. We assume g to be an odd function and conclude that

$$g = Cy \quad \text{and} \quad f = \sqrt{C^{-1}x + B}, \tag{116}$$

where B is constant and we take $C > 0$ without loss of generality. We consider two cases:

- $\boldsymbol{D} = \boldsymbol{i}$
- $\boldsymbol{D} = \boldsymbol{j}$

An Example: Flexure, First Case $D = i$

The fibers are initially horizontal and mapped by the deformation to rays through the origin. From $\lambda \boldsymbol{d} = \boldsymbol{F}\boldsymbol{D}$

$$\lambda \boldsymbol{d} = \boldsymbol{F}\boldsymbol{i} = f'(x)\boldsymbol{e}_r \Longrightarrow \boldsymbol{d} = \boldsymbol{e}_r(\theta) \quad \text{and} \quad \lambda = f'(x). \tag{117}$$

Then,

$$\boldsymbol{d}' = \boldsymbol{d}_{,x} = \boldsymbol{e}_\theta \theta_{,x} = 0 \qquad \text{(because } \theta \text{ is a function of } y \text{ alone)} \tag{118}$$

Accordingly, the fibers remain straight in the course of the deformation, thus

$$\boldsymbol{m} = T\gamma \boldsymbol{d} + F\boldsymbol{d} \times \boldsymbol{d}' \longrightarrow \boldsymbol{m} = T\gamma \boldsymbol{e}_r. \tag{119}$$

also

$$\boldsymbol{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} = 0 \xrightarrow{\cdot \boldsymbol{d}} 0 = \boldsymbol{d} \cdot \boldsymbol{m}' = \boldsymbol{e}_r \cdot \boldsymbol{m}_{,x} = T\gamma_{,x} \Longrightarrow \gamma'(= \gamma_{,x}) = 0 \tag{120}$$

This in turn implies

$$\boldsymbol{m}' = 0 \longrightarrow \boldsymbol{d} \times \boldsymbol{\lambda} = 0 \xrightarrow{\text{because } \boldsymbol{d} \cdot \boldsymbol{\lambda} = 0} \boldsymbol{\lambda} = 0 \tag{121}$$

Further, if there are no twisting couples at the vertical boundaries $x = const.$, then γ and $\boldsymbol{m}$ vanish everywhere; the fibers are effectively inactive. Of course there exists an equilibrating pressure field because this deformation is controllable.

The fibers are initially vertical and hence mapped by the deformation to concentric circles. In place of $\lambda \boldsymbol{d} = \boldsymbol{Fi} = f'(x)\boldsymbol{e}_r$ we have

$$\lambda \boldsymbol{d} = \boldsymbol{Fj} = fg'\boldsymbol{e}_\theta \longrightarrow \boldsymbol{d} = \boldsymbol{e}_\theta \quad \text{and} \quad \lambda = fg' = C\sqrt{C^{-1}x + B}. \tag{122}$$

Then, $\boldsymbol{d}' = \boldsymbol{d}_{,y} = \boldsymbol{e}_\theta' g'(y) = -C\boldsymbol{e}_r$.

Also

$$\boldsymbol{m} = T\gamma \boldsymbol{d} + F\boldsymbol{d} \times \boldsymbol{d}' \Longrightarrow \boldsymbol{m} = T\gamma \boldsymbol{e}_\theta + FC\boldsymbol{k}. \tag{123}$$

Moreover

$$\boldsymbol{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} = 0 \xrightarrow{\cdot \boldsymbol{d}} 0 = \boldsymbol{d} \cdot \boldsymbol{m}' = \boldsymbol{e}_\theta \cdot \boldsymbol{m}_{,y} = T\gamma_{,y} \Longrightarrow \gamma'(= \gamma_{,y}) = 0 \tag{124}$$

If no twisting couples are applied at the horizontal boundaries $y = const.$, then the fiber twist vanishes everywhere and

$$\boldsymbol{m} = FC\boldsymbol{k} \longrightarrow \boldsymbol{m}' = 0 \Longrightarrow \boldsymbol{\lambda} = 0 \tag{125}$$

as before. The moment $\boldsymbol{m}$ combines with the overall bending moment generated by the matrix material.

An Example: Bending, stretching and shearing of a block

This is another controllable deformation, obtained by composing transverse shear with flexure. First we deform the block by flexure to the configuration defined by $\boldsymbol{x}_1 = \chi_1(\boldsymbol{X})$, where χ_1 is given by $\chi_1(\boldsymbol{X}) = r\boldsymbol{e}_r(\theta) + z\boldsymbol{k}$. Then the block is sheared to the configuration

$$\boldsymbol{x}_2 = \chi_2(\boldsymbol{x}_1) = r\boldsymbol{e}_r(\theta) + \varsigma\boldsymbol{k}, \quad \text{where} \quad \varsigma = z + \beta\theta, \tag{126}$$

with β a positive constant. This maps a plane $z = const.$ to a helicoidal surface. We obtain $\boldsymbol{F} = \boldsymbol{F}_2\boldsymbol{F}_1$ with

$$\boldsymbol{F}_1 = f'\boldsymbol{e}_r \otimes \boldsymbol{i} + fg'\boldsymbol{e}_\theta \otimes \boldsymbol{j} + \boldsymbol{k} \otimes \boldsymbol{k} \qquad \text{and} \qquad \boldsymbol{F}_2 = \boldsymbol{I} + \beta r^{-1}\boldsymbol{k} \otimes \boldsymbol{e}_\theta. \tag{127}$$

For the case $\boldsymbol{D} = \boldsymbol{j}$ we compute

$$\lambda\boldsymbol{d} = \boldsymbol{F}\boldsymbol{j} = C(r\boldsymbol{e}_\theta + \beta\boldsymbol{k}) \longrightarrow \lambda = C\sqrt{r^2 + \beta^2} \quad \text{(a function of } x \text{ alone)} \tag{128}$$

The fibers $r = const.$ are circular helices of constant pitch. From these results we find

$$\boldsymbol{d}' = \boldsymbol{d}_{,y} = \frac{-Cr}{\sqrt{r^2 + \beta^2}}\boldsymbol{e}_r(\theta) \quad \text{and} \quad \boldsymbol{d} \times \boldsymbol{d}' = \frac{Cr^2}{r^2 + \beta^2}(\boldsymbol{k} - \beta r^{-1}\boldsymbol{e}_\theta). \tag{129}$$

An Example: Bending, stretching and shearing of a block [4]

Using $\boldsymbol{m} = T\gamma\boldsymbol{d} + F\boldsymbol{d} \times \boldsymbol{d}'$ and $\boldsymbol{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} = 0$ we again obtain $\gamma'(= \gamma_{,y}) = 0$ and conclude that the fiber twist is a function of x alone. To determine it we substitute into $\boldsymbol{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} = 0$, obtaining

$$T\gamma\boldsymbol{d}' + F\boldsymbol{d} \times \boldsymbol{d}'' = \lambda\boldsymbol{\lambda} \times \boldsymbol{d}, \tag{130}$$

and hence

$$\boldsymbol{\lambda} \times \boldsymbol{d} = \lambda^{-1}\left(F\frac{C^2\beta r}{r^2 + \beta^2} - T\frac{\gamma C r}{\sqrt{r^2 + \beta^2}}\right)\boldsymbol{e}_r. \tag{131}$$

This yields $\boldsymbol{\lambda} = \boldsymbol{d} \times (\boldsymbol{\lambda} \times \boldsymbol{d})$ in terms of $\gamma(x)$.

A force-free solution ($\boldsymbol{\lambda} = 0$) is available, with fiber twist

$$\gamma(x) = \frac{F}{T}\frac{C\beta}{\sqrt{r^2 + \beta^2}}, \quad \text{where} \quad r = f(x). \tag{132}$$

Then $\boldsymbol{m}$ is fully determined by the constitutive equation $\boldsymbol{m} = T\gamma\boldsymbol{d} + F\boldsymbol{d} \times \boldsymbol{d}'$, and includes both bending and twisting components. We have satisfied

$$\boldsymbol{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} = 0, \qquad \boldsymbol{\lambda}' = 0 \tag{133}$$

and so $\boldsymbol{\lambda}' + \mu div\boldsymbol{B} = \mathrm{grad} p$ yields the existence of an equilibrating pressure field by virtue of the controllability of the deformation.

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thank you!